

Problems with Point Interactions

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Abstract

The integrability of one dimensional quantum mechanical many-body problems with general contact interactions is extensively studied. It is shown that besides the pure (repulsive or attractive) δ -function interaction there is another singular point interactions which gives rise to a new one-parameter family of integrable quantum mechanical many-body systems. The bound states and scattering matrices are calculated for both bosonic and fermionic statistics.

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Quantum mechanical solvable models describing a particle moving in a local singular potential concentrated at one or a discrete number of points have been extensively discussed in the literature, see e.g. [1, 2, 3] and references therein. One dimensional problems with contact interactions at, say, the origin ($x = 0$) can be characterized by separated or nonseparated boundary conditions imposed on the (scalar) wave function φ at $x = 0$. The classification of one dimensional point interactions in terms of singular perturbations is given in [4]. In the present paper we are interested in many-body problems with pairwise interactions given by such singular potentials. The first model of this type with the pairwise interactions determined by δ -functions was suggested and investigated in [5]. Intensive studies of this model applied to statistical mechanics (particles having boson or fermion statistics) are given in [6, 7] (these also leads to the well known Yang-Baxter equations).

Nonseparated boundary conditions correspond to the cases where the perturbed operator is equal to the orthogonal sum of two self-adjoint operators in $L_2(-\infty, 0]$ and $L_2[0, \infty)$. The family of point interactions for the one dimensional Schrödinger operator $-\frac{d^2}{dx^2}$ can be described by unitary 2×2 matrices via von Neumann formulas for self-adjoint extensions of symmetric operators, since the second derivative operator restricted to the domain $C_0^\infty(\mathbf{R} \setminus \{0\})$ has deficiency indices $(2, 2)$. The boundary conditions describing the self-adjoint extensions have the following form

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}_{0+} = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}_{0-}, \quad (1)$$

where

$$ad - bc = 1, \quad \theta, a, b, c, d \in \mathbb{R}. \quad (2)$$

$\varphi(x)$ is the scalar wave function of two spinless particles with relative coordinate x . (1) also describes two particles with spin s but without any spin coupling between the particles when they meet (i.e. for $x = 0$), in this case φ represents any one of the components of the wave function. The values $\theta = b = 0$, $a = d = 1$ in (1) correspond to the case of a positive (resp. negative) δ -function potential for $c > 0$ (resp. $c < 0$). For general

a, b, c and d , the properties of the corresponding Hamiltonian systems have been studied in detail, see e.g. [4, 8, 9].

The separated boundary conditions are described by

$$\varphi'(0_+) = h^+ \varphi(0_+) , \quad \varphi'(0_-) = h^- \varphi(0_-), \quad (3)$$

where $h^\pm \in \mathbb{R} \cup \{\infty\}$. $h^+ = \infty$ or $h^- = \infty$ correspond to Dirichlet boundary conditions and $h^+ = 0$ or $h^- = 0$ correspond to Neumann boundary conditions. In this case it is impossible to express the perturbed operator as the orthogonal sum of two self-adjoint operators in $L_2(-\infty, 0]$ and $L_2[0, \infty)$.

In the following we study the integrability of one dimensional systems of N -identical particles with general contact interactions described by the boundary conditions (1) or (3) that are imposed on the relative coordinates of the particles. We first consider the case of two particles ($N = 2$) with coordinates x_1, x_2 and momenta k_1, k_2 respectively. Each particle has n -‘spin’ states designated by s_1 and s_2 , $1 \leq s_i \leq n$. For $x_1 \neq x_2$, these two particles are free. The wave functions φ are symmetric (resp. antisymmetric) with respect to the interchange $(x_1, s_1) \leftrightarrow (x_2, s_2)$ for bosons (resp. fermions). In the region $x_1 < x_2$, from the Bethe ansatz the wave function is of the form,

$$\varphi = \alpha_{12} e^{i(k_1 x_1 + k_2 x_2)} + \alpha_{21} e^{i(k_2 x_1 + k_1 x_2)}, \quad (4)$$

where α_{12} and α_{21} are $n^2 \times 1$ column matrices. In the region $x_1 > x_2$,

$$\varphi = (P^{12} \alpha_{12}) e^{i(k_1 x_2 + k_2 x_1)} + (P^{12} \alpha_{21}) e^{i(k_2 x_2 + k_1 x_1)}, \quad (5)$$

where according to the symmetry or antisymmetry conditions, $P^{12} = p^{12}$ for bosons and $P^{12} = -p^{12}$ for fermions, p^{12} being the operator on the $n^2 \times 1$ column that interchanges $s_1 \leftrightarrow s_2$.

Let $k_{12} = (k_1 - k_2)/2$. In the center of mass coordinate $X = (x_1 + x_2)/2$ and the relative coordinate $x = x_2 - x_1$, we get, by substituting (4) and (5) into the boundary conditions at $x = 0$,

$$\begin{cases} \alpha_{12} + \alpha_{21} = e^{i\theta} a P^{12} (\alpha_{12} + \alpha_{21}) + i e^{i\theta} b k_{12} P^{12} (\alpha_{12} - \alpha_{21}), \\ i k_{12} (\alpha_{21} - \alpha_{12}) = e^{i\theta} c P^{12} (\alpha_{12} + \alpha_{21}) + i e^{i\theta} d k_{12} P^{12} (\alpha_{12} - \alpha_{21}) \end{cases} \quad (6)$$

for boundary condition (1), and

$$\begin{cases} ik_{12}(\alpha_{21} - \alpha_{12}) = h_+(\alpha_{12} + \alpha_{21}) , \\ ik_{12}P^{12}(\alpha_{12} - \alpha_{21}) = h_-P^{12}(\alpha_{12} + \alpha_{21}) \end{cases} \quad (7)$$

for boundary condition (3) respectively.

Eliminating the term $P^{12}\alpha_{12}$ from (6) we obtain the relation

$$\alpha_{21} = Y_{21}^{12} \alpha_{12} , \quad (8)$$

where

$$Y_{21}^{12} = \frac{2ie^{i\theta}k_{12}P^{12} + ik_{12}(a - d) + (k_{12})^2b + c}{ik_{12}(a + d) + (k_{12})^2b - c}. \quad (9)$$

We remark that the system (7) is contradictory unless

$$h_+ = -h_- \doteq h \in \mathbb{R} \cup \{\infty\}. \quad (10)$$

In this case it also leads to equation (8) with

$$Y_{21}^{12} = \frac{ik_{12} + h}{ik_{12} - h}. \quad (11)$$

For $N \geq 3$ and $x_1 < x_2 < \dots < x_N$, the wave function is given by

$$\psi = \alpha_{12\dots N} e^{i(k_1x_1 + k_2x_2 + \dots + k_Nx_N)} + \alpha_{21\dots N} e^{i(k_2x_1 + k_1x_2 + \dots + k_Nx_N)} + (N! - 2) \text{ other terms}. \quad (12)$$

The columns α have $n^N \times 1$ dimensions. The wave functions in the other regions are determined from (12) by the requirement of symmetry (for bosons) or antisymmetry (for fermions). Along any plane $x_i = x_{i+1}$, $i \in 1, 2, \dots, N - 1$, from similar considerations as above we have

$$\alpha_{l_1 l_2 \dots l_i l_{i+1} \dots l_N} = Y_{l_{i+1} l_i}^{ii+1} \alpha_{l_1 l_2 \dots l_{i+1} l_i \dots l_N}, \quad (13)$$

where

$$Y_{l_{i+1} l_i}^{ii+1} = \frac{2ie^{i\theta}k_{l_i l_{i+1}}P^{ii+1} + ik_{l_i l_{i+1}}(a - d) + (k_{l_i l_{i+1}})^2b + c}{ik_{l_i l_{i+1}}(a + d) + (k_{l_i l_{i+1}})^2b - c} \quad (14)$$

for nonseparated boundary condition and

$$Y_{l_{i+1} l_i}^{ii+1} = \frac{ik_{l_i l_{i+1}} + h}{ik_{l_i l_{i+1}} - h} \quad (15)$$

for separated boundary condition. Here $k_{l_i l_{i+1}} = (k_{l_i} - k_{l_{i+1}})/2$ play the role of spectral parameters. $P^{ii+1} = p^{ii+1}$ for bosons and $P^{ii+1} = -p^{ii+1}$ for fermions, with p^{ii+1} the operator on the $n^N \times 1$ column that interchanges $s_i \leftrightarrow s_{i+1}$.

For consistency Y must satisfy the Yang-Baxter equation with spectral parameter [6, 10], i.e.,

$$Y_{ij}^{m,m+1} Y_{kj}^{m+1,m+2} Y_{ki}^{m,m+1} = Y_{ki}^{m+1,m+2} Y_{kj}^{m,m+1} Y_{ij}^{m+1,m+2},$$

or

$$Y_{ij}^{mr} Y_{kj}^{rs} Y_{ki}^{mr} = Y_{ki}^{rs} Y_{kj}^{mr} Y_{ij}^{rs} \quad (16)$$

if m, r, s are all unequal, and

$$Y_{ij}^{mr} Y_{ji}^{mr} = 1, \quad Y_{ij}^{mr} Y_{kl}^{sq} = Y_{kl}^{sq} Y_{ij}^{mr} \quad (17)$$

if m, r, s, q are all unequal.

The operators Y given by (14) satisfy the relation (17) for all θ, a, b, c, d . However the relations (16) are satisfied only when $\theta = 0$, $a = d$ and $b = 0$, that is, according to the constraint (2), $\theta = 0$, $a = d = \pm 1$, $b = 0$, c arbitrary. The case $a = d = 1$, $\theta = b = 0$ corresponds to the usual δ -function interactions, which has been investigated in [6, 7]. The case $a = d = -1$, $\theta = b = 0$, which we shall refer to as ‘anti- δ ’ interaction, is related to another singular interactions between any pair of particles (for $a = d = -1$ and $\theta = b = c = 0$ see [4, 8]). Associated with the separated boundary condition, the operators Y given by (15) satisfy both the relations (17) and (16) for arbitrary h .

We have thus found that with respect to N -particle (either boson or fermion) problems, altogether there are three integrable one parameter families with contact interactions of type δ , anti- δ and separated one, described respectively by one of the following conditions on the wave function along the plane $x_i = x_j$ for any pair of particles with coordinates x_i and x_j ,

$$\varphi(0_+) = +\varphi(0_-), \quad \varphi'(0_+) = c\varphi(0_-) + \varphi'(0_-), \quad c \in \mathbb{R}; \quad (18)$$

$$\varphi(0_+) = -\varphi(0_-), \quad \varphi'(0_+) = c\varphi(0_-) - \varphi'(0_-), \quad c \in \mathbb{R}; \quad (19)$$

$$\varphi'(0_+) = h\varphi(0_+), \quad \varphi'(0_-) = -h\varphi(0_-), \quad h \in \mathbb{R} \cup \{\infty\}. \quad (20)$$

The wave functions are given by (12) with the α 's determined by (13) and initial conditions. The operators Y in (13) are given respectively by

$$Y_{l_{i+1}l_i}^{ii+1} = \frac{i(k_{l_i} - k_{l_{i+1}})P^{ii+1} + c}{i(k_{l_i} - k_{l_{i+1}}) - c}; \quad (21)$$

$$Y_{l_{i+1}l_i}^{ii+1} = -\frac{i(k_{l_i} - k_{l_{i+1}})P^{ii+1} + c}{i(k_{l_i} - k_{l_{i+1}}) + c}; \quad (22)$$

and

$$Y_{l_{i+1}l_i}^{ii+1} = \frac{i(k_{l_i} - k_{l_{i+1}}) + 2h}{i(k_{l_i} - k_{l_{i+1}}) - 2h}. \quad (23)$$

Nevertheless, from (21) and (22) we see that if we simultaneously change $c \rightarrow -c$ and $P^{ii+1} \rightarrow -P^{ii+1}$, these two formulas are interchanged. There is a sort of duality between bosons (resp. fermions) with δ -interaction of strength c and fermions (resp. bosons) with anti- δ interaction of strength $-c$. It can be checked that under the “kink type” gauge transformation $\mathcal{U} = \prod_{i>j} \text{sgn}(x_i - x_j)$, the N-boson (resp. fermion) δ -type contact interaction goes over to the N-fermion (resp. boson) anti- δ interaction. Therefore these two situations are in fact unitarily equivalent under a gauge transformation \mathcal{U} that is non-smooth and does not factorize through one particle Hilbert spaces.

The integrable system related to the case (23) is not unitarily equivalent to either the δ or anti- δ cases. In fact their spectra are different (see the bound states below). In the following we study further the one dimensional integrable N -particle systems associated with (23).

When $h < 0$, there exist bound states. For $N = 2$, the space part of the orthogonal basis (labeled by \pm) in the doubly degenerate bound state subspace has the form, in the relative coordinate $x = x_2 - x_1$,

$$\psi_{2,\pm} = (\theta(x) \pm \theta(-x))e^{h|x|}. \quad (24)$$

The eigenvalue corresponding to the bound states (24) is $-h^2$. By generalization we get the $2^{N(N-1)/2}$ bound states for N -particle system

$$\psi_{N,\underline{\epsilon}} = \alpha_{\underline{\epsilon}} \prod_{k>l} (\theta(x_k - x_l) + \epsilon_{kl}\theta(x_l - x_k)) e^{h \sum_{i>j} |x_i - x_j|}, \quad (25)$$

where $\alpha_{\underline{\epsilon}}$ is the spin wave function and $\underline{\epsilon} \equiv \{\epsilon_{kl} : k > l\}$; $\epsilon_{kl} = \pm$, labels the $2^{N(N-1)/2}$ -fold degeneracy.

It can be checked that $\psi_{N,\underline{\epsilon}}$ satisfies the boundary condition (20) at $x_i = x_j$ for any $i \neq j \in 1, \dots, N$. The spin wave function α here satisfies $P^{ij}\alpha = \epsilon_{ij}\alpha$ for any $i \neq j$, that is, $p^{ij}\alpha = \epsilon_{ij}\alpha$ for bosons and $p^{ij}\alpha = -\epsilon_{ij}\alpha$ for fermions. $\psi_{N,\underline{\epsilon}}$ is of the form (12) in each region. For instance comparing $\psi_{N,\underline{\epsilon}}$ with (12) in the region $x_1 < x_2 \dots < x_N$, we get

$$k_1 = ih(N-1), \quad k_2 = k_1 - 2ih, \quad k_3 = k_2 - 2ih, \dots, k_N = -k_1. \quad (26)$$

The energy of the bound state $\psi_{N,\underline{\epsilon}}$ is

$$E = -\frac{h^2}{3}N(N^2 - 1). \quad (27)$$

The scattering matrix can readily be discussed. For real $k_1 < k_2 < \dots k_N$, in each coordinate region such as $x_1 < x_2 < \dots x_N$, the following term in (12) is an outgoing wave

$$\psi_{out} = \alpha_{12\dots N} e^{k_1 x_1 + \dots + k_N x_N}. \quad (28)$$

An incoming wave with the same exponential as (28) is given by

$$\psi_{in} = [P^{1N} P^{2(N-1)} \dots] \alpha_{N(N-1)\dots 1} e^{k_N x_N + \dots + k_1 x_1} \quad (29)$$

in the region $x_N < x_{N-1} < \dots < x_1$. The scattering matrix is defined by $\psi_{out} = S\psi_{in}$.

From (13) we have

$$\begin{aligned} \alpha_{12\dots N} &= [Y_{21}^{12} Y_{31}^{23} \dots Y_{N1}^{(N-1)N}] \alpha_{2\dots N1} = \dots \\ &= [Y_{21}^{12} Y_{31}^{23} \dots Y_{N1}^{(N-1)N}] [Y_{32}^{12} Y_{42}^{23} \dots Y_{N2}^{(N-2)(N-1)}] \dots [Y_{N(N-1)}^{12}] \alpha_{N(N-1)\dots 1} \equiv S' \alpha_{N(N-1)\dots 1}, \end{aligned}$$

where $Y_{l_{i+1}l_i}^{i+1}$ is given by (23). Therefore

$$S = S' P^{N1} P^{(N-1)2} \dots P^{1N} = S' [P^{12}] [P^{23} P^{12}] [P^{34} P^{23} P^{12}] \dots [P^{(N-1)N} \dots P^{12}].$$

Defining

$$X_{ij} = Y_{ij}^{ij} P^{ij} \quad (30)$$

we obtain

$$S = [X_{21} X_{31} \dots X_{N1}] [X_{32} X_{42} \dots X_{N2}] \dots [X_{N(N-1)}]. \quad (31)$$

The scattering matrix S is unitary and symmetric due to the time reversal invariance of the interactions. $\langle s'_1 s'_2 \dots s'_N | S | s_1 s_2 \dots s_N \rangle$ stands for the S matrix element of the process from the state $(k_1 s_1, k_2 s_2, \dots, k_N s_N)$ to the state $(k_1 s'_1, k_2 s'_2, \dots, k_N s'_N)$. The momenta (26) are imaginary for bound states. The scattering of clusters (bound states) can be discussed in a similar way as in [7]. For instance for the scattering of a bound state of two particles $(x_1 < x_2)$ on a bound state of three particles $(x_3 < x_4 < x_5)$, the scattering matrix is $S = [X_{32} X_{42} X_{52}] [X_{31} X_{41} X_{51}]$.

We have extensively investigated the integrability of one dimensional quantum mechanical many-body problems with general contact interactions. Besides the repulsive or attractive δ and anti- δ function interactions, there is another integrable one parameter families associated with separated boundary conditions. From our calculations it is clear that these are all the integrable systems for one dimensional quantum identical many-particle models (of fermionic or bosonic statistics) with contact interactions. Here the possible contact coupling of the spins of two particles are not taken into account. A further study along this direction would possibly give rise to more interesting integrable quantum many-body systems.

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